Prove that $\frac{d}{dt}[(\vec{r}(t)\times\vec{r}'(t))\cdot\vec{r}''(t)] = (\vec{r}(t)\times\vec{r}'(t))\cdot\vec{r}'''(t)$ using Section 13.2 Theorem 3 **SCORE: _____ / 6 PTS**
and the properties of the dot and cross products.

$$\frac{d}{dt}[(\vec{r}(t)\times\vec{r}'(t))\cdot\vec{r}''(t)]$$

$$=\left[\frac{d}{dt}(\vec{r}(t)\times\vec{r}'(t))\right]\cdot\vec{r}''(t)+\left[\vec{r}(t)\times\vec{r}'(t)\right]\cdot\vec{r}'''(t)$$

$$=\left[\vec{r}'(t)\times\vec{r}'(t)+\vec{r}(t)\times\vec{r}''(t)\right]\cdot\vec{r}''(t)+\left[\vec{r}(t)\times\vec{r}'(t)\right]\cdot\vec{r}'''(t)$$

$$=\left[\vec{0}+\vec{r}(t)\times\vec{r}''(t)\right]\cdot\vec{r}''(t)+\left[\vec{r}(t)\times\vec{r}'(t)\right]\cdot\vec{r}'''(t)$$

$$=\vec{r}(t)\cdot\left[\vec{r}''(t)\times\vec{r}''(t)\right]+\left[\vec{r}(t)\times\vec{r}'(t)\right]\cdot\vec{r}'''(t)$$

$$=\vec{r}(t)\cdot\vec{0}+\left[\vec{r}(t)\times\vec{r}'(t)\right]\cdot\vec{r}'''(t)$$

$$=(\vec{r}(t)\times\vec{r}'(t))\cdot\vec{r}'''(t)$$

Find the curvature function $\kappa(t)$ for the vector function $\vec{r}(t)=<1-2t-t^2, t^2-t-2, t^2+2t-1>$. **SCORE: _____ / 6 PTS**
NOTE: There are 2 ways to compute this, one of which is MUCH longer than the other.

$$\vec{r}'(t)=<-2-2t, 2t-1, 2t+2>$$

$$\vec{r}''(t)=<-2, 2, 2>=2<-1,1,1>$$

$$\vec{r}'(t)\times\vec{r}''(t)=<-6, 0, -6>=-6<1,0,1>$$

$$\kappa(t)=\frac{\|\vec{r}'(t)\times\vec{r}''(t)\|}{\|\vec{r}'(t)\|^3}=\frac{6\sqrt{2}}{(\sqrt{(-2-2t)^2+(2t-1)^2+(2t+2)^2})^3}=\frac{6\sqrt{2}}{(\sqrt{9+12t+12t^2})^3}$$

Consider the vector function $\vec{r}(t) = < e^{-t}\sin 2t, 1-2e^{-t}, e^{-t}\cos 2t >$ and the corresponding space curve defined by $\vec{r}(t)$.

SCORE: _____ / 18 PTS

[a] Find $\vec{r}\,'(0)$.

$$\vec{r}\,'(t) = < -e^{-t}\sin 2t + 2e^{-t}\cos 2t, 2e^{-t}, -e^{-t}\cos 2t - 2e^{-t}\sin 2t >$$
$$= -e^{-t} < \sin 2t - 2\cos 2t, -2, \cos 2t + 2\sin 2t >$$
$$\vec{r}\,'(0) = < 2, 2, -1 >$$

[b] Find a symmetric equation for the tangent line to the curve at the point where $t = 0$.

$$\vec{r}(0) = < 0, -1, 1 >$$
$$\frac{x}{2} = \frac{y+1}{2} = 1 - z$$

[c] Find $\vec{T}(t)$.

$$\|\vec{r}\,'(t)\| = |-e^{-t}|\sqrt{(\sin 2t - 2\cos 2t)^2 + (-2)^2 + (\cos 2t + 2\sin 2t)^2}$$
$$= e^{-t}\sqrt{(\sin^2 2t - 4\sin 2t\cos 2t + 4\cos^2 2t) + 4 + (\cos^2 2t + 4\cos 2t\sin 2t + 4\sin^2 2t)}$$
$$= 3e^{-t}$$
$$\vec{T}(t) = \frac{\vec{r}\,'(t)}{\|\vec{r}\,'(t)\|} = -\tfrac{1}{3} < \sin 2t - 2\cos 2t, -2, \cos 2t + 2\sin 2t >$$

[d] Find the length of the curve from the point where $t = 0$ to the point where $t = \pi$.

$$s(\pi) = \int_0^{\pi} \|\vec{r}\,'(t)\|\,dt = \int_0^{\pi} 3e^{-t}\,dt = -3e^{-t}\Big|_0^{\pi} = -3(e^{-\pi} - 1) = 3(1 - e^{-\pi})$$

[e] Reparametrize the curve with respect to arc length measured from the point where $t = 0$ in the direction of increasing t. Simplify your final answer completely.

$$s(t) = \int_0^{t} \|\vec{r}\,'(u)\|\,du = 3(1 - e^{-t})$$
$$e^{-t} = 1 - \tfrac{1}{3}s \Rightarrow t = -\ln(1 - \tfrac{1}{3}s)$$
$$\vec{r}(s) = < (1 - \tfrac{1}{3}s)\sin 2(-\ln(1 - \tfrac{1}{3}s)), 1 - 2(1 - \tfrac{1}{3}s), (1 - \tfrac{1}{3}s)\cos 2(-\ln(1 - \tfrac{1}{3}s)) >$$
$$= < (\tfrac{1}{3}s - 1)\sin(2\ln(1 - \tfrac{1}{3}s)), \tfrac{2}{3}s - 1, (1 - \tfrac{1}{3}s)\cos(2\ln(1 - \tfrac{1}{3}s)) >$$

[f] Find the curvature function $\kappa(t)$. **<u>NOTE: There are 2 ways to compute this, one of which is MUCH longer than the other.</u>**

$$\vec{T}\,'(t) = -\tfrac{1}{3} < 2\cos 2t + 4\sin 2t, 0, -2\sin 2t + 4\cos 2t > = -\tfrac{2}{3} < \cos 2t + 2\sin 2t, 0, -\sin 2t + 2\cos 2t >$$
$$\|\vec{T}\,'(t)\| = |-\tfrac{2}{3}|\sqrt{(\cos 2t + 2\sin 2t)^2 + (-\sin 2t + 2\cos 2t)^2}$$
$$= \tfrac{2}{3}\sqrt{(\cos^2 2t + 4\cos 2t\sin 2t + 4\sin^2 2t) + (\sin^2 2t - 4\sin 2t\cos 2t + 4\cos^2 2t)} = \tfrac{2\sqrt{5}}{3}$$
$$\kappa(t) = \frac{\|\vec{T}\,'(t)\|}{\|\vec{r}\,'(t)\|} = \frac{\frac{2\sqrt{5}}{3}}{3e^{-t}} = \tfrac{2\sqrt{5}}{9}e^{t}$$